the indices of the face in question. The indices of a prism face like $l(310)$ can be readily obtained in exactly the same manner as described under the Isometric System, Art. 84. p. 75.

III. HEXAGONAL SYSTEM

119. The Hexagonal System includes all the forms which are referred to four axes, three equal horizontal axes in a common plane intersecting at angles of 60°, and a fourth, vertical axis, at right angles to them.

Two sections are here included, each embracing a number of distinct classes related among themselves. They are called the *Hexagonal Division* and the *Trigonal* (or *Rhombohedral*) *Division*. The symmetry of the former, about the vertical axis, belongs to the hexagonal type, that of the latter to the trigonal type.

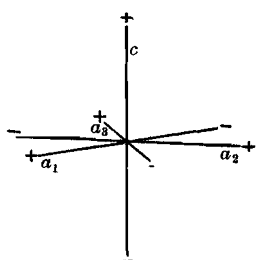
Miller (1852) referred all the forms of the hexagonal system to three equal axes parallel to the faces of the fundamental rhombohedron, and hence intersecting at equal angles, not 90°. This method (further explained in Art. **169**) had the disadvantage of failing to bring out the relationship between the normal hexagonal and tetragonal types, both characterized by a principal axis of symmetry, which (on the system adopted in this book) is the vertical crystallographic axis. It further gave different symbols to faces which are crystallographically identical. It is more natural to employ the three rhombohedral axes for trigonal forms only, as done by Groth (1905), who includes these groups in a *Trigonal System;* but this also has some disadvantages. The indices commonly used in describing hexagonal forms are known as the Miller-Bravais indices, since they were adopted by Bravais for use with the four axes from the scheme used by Miller in the other crystal systems.

120. Symmetry Classes. — There are five possible classes in the Hexagonal Division. Of these the normal class is much the most important, and two others are also of importance among crystallized minerals.

In the Trigonal Division there are seven classes; of these the rhombohedral class or that of the Calcite Type, is by far the most common, and three others are also of importance.

121. Axes and Symbols. — The position of the four axes taken is shown in Fig. 218; the three horizontal axes are called a, since they are equal and interchangeable, and the vertical axis is c, since it has a different length, being either longer or shorter than the horizontal axes. The length of the vertical axis is expressed in terms of that of the horizontal axes which in turn is always taken as unity. Further, when it is desirable to distinguish between the horizontal axes they may be designated a_1, a_2, a_3. When properly orientated one of the horizontal axes (a_2) is parallel to the observer and the other two make angles of 30° either side of the line perpendicular to him. The axis to the left is taken as a_1, the one to the right as a_3. The positive and negative ends of the axes are shown in Fig. 218. The general position of any plane may be expressed in a manner analogous to that applicable in the other systems, viz.

218



Hexagonal Axes

$$\frac{1}{h}a_1 : \frac{1}{k}a_2 : \frac{1}{i}a_3 : \frac{1}{l}c.$$

The corresponding indices for a given plane are then h, k, i, l; these always refer to the axes named in the above scheme. Since it is found convenient to consider the axis a_3 as negative in front and positive behind, the general symbol becomes $hk\bar{i}l$. Further, as following from the angular relation of the three horizontal axes, it can be readily shown to be always true that the algebraic sum of the indices h, k, i, is equal to zero:

$$h + k + i = 0.$$

A. Hexagonal Division

1. NORMAL CLASS (13). BERYL TYPE

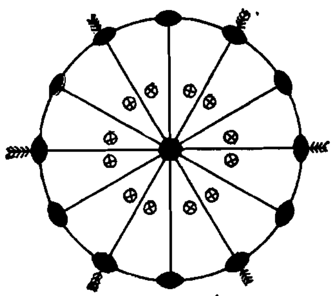
(Dihexagonal Bipyramidal or Holohedral Class)

122. Symmetry. — Crystals belonging to the normal class of the Hexagonal Division have one principal axis of hexagonal, or sixfold, symmetry, which coincides with the vertical crystallographic axis; also six horizontal axes of binary symmetry; three of these coincide with the horizontal crystallographic axes, the others bisect the angles between them. There is one principal plane of symmetry which is the plane of the horizontal crystallographic axes and six vertical planes of symmetry which meet in the vertical crystallographic axis. Three of these vertical planes include the horizontal crystallographic axes and the other three bisect the angles between the first set.

The symmetry of this class is exhibited in the accompanying stereographic projection, Fig. 219, and by the following crystal figures.

The analogy between this class and the normal class of the tetragonal system is obvious at once and will be better appreciated as greater familiarity is gained with the individual forms and their combinations.

219



Symmetry of Normal Class

123. Forms. — The possible forms in this class are as follows:

	Miller-Bravais.
1. Base	(0001)
2. Prism of the first order	(1010)
3. Prism of the second order	(1120)
4. Dihexagonal prism	$(hk\bar{i}0)$ as, $(21\bar{3}0)$
5. Pyramid of the first order	$(h0\bar{h}l)$ as, $(10\bar{1}1)$; $(20\bar{2}1)$ etc.
6. Pyramid of the second order	$(h \cdot h \cdot 2\bar{h} \cdot l)$ as, $(11\bar{2}2)$
7. Dihexagonal pyramid	$(hkil)$ as, $(21\bar{3}1)$

In the above $h > k$, and $h + k = -i$.

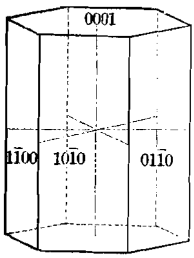
124. Base. — The *base*, or *basal pinacoid*, includes the two faces, 0001 and $000\bar{1}$, parallel to the plane of the horizontal axes. It is uniformly designated by the letter c; see Fig. 220 *et seq.*

125. Prism of the First Order. — There are three types of prisms, or forms in which the faces are parallel to the vertical axis.

The *prism of the first order*, Fig. 220, includes six faces, each one of which is parallel to the vertical axis and meets two adjacent horizontal axes at equal distances, while it is parallel to the third horizontal axis. It has hence the general symbol $(10\bar{1}0)$ and is uniformly designated by the letter m; the indices of its six faces taken in order (see Figs. 220 and 229, 230) are:

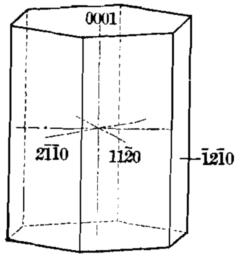
$$10\bar{1}0,\quad 01\bar{1}0,\quad \bar{1}100,\quad \bar{1}010,\quad 0\bar{1}10,\quad 1\bar{1}00.$$

220



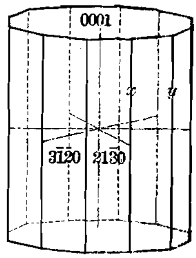
First Order Prism

221



Second Order Prism

222



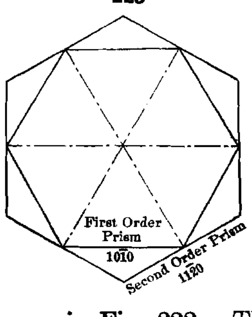
Dihexagonal Prism

126. Prism of the Second Order. — The *prism of the second order*, Fig. 221, has six faces, each one of which is parallel to the vertical axis, and meets the three horizontal axes, two alternate axes at the unit distance, the intermediate axis at one-half this distance; or, which is the same thing, it meets the last-named axis at the unit distance, the others at double this distance.* The general symbol is $(11\bar{2}0)$ and it is uniformly designated by the letter a; the indices of the six faces (see Figs. 221 and 229, 230) in order are:

$$11\bar{2}0,\quad \bar{1}2\bar{1}0,\quad \bar{2}110,\quad \bar{1}\bar{1}20,\quad 1\bar{2}10,\quad 2\bar{1}\bar{1}0.$$

The first and second order prisms are not to be distinguished geometrically from each other since each is a regular hexagonal prism with normal interfacial angles of 60°. They are related to each other in the same way as the two prisms $m(110)$ and $a(100)$ of the tetragonal system.

223



The relation in position between the first order prism (and pyramids) on the one hand and the second order prism (and pyramids) on the other will be understood better from Fig. 223, representing a cross section of the two prisms parallel to the base c.

127. Dihexagonal Prism. — The *dihexagonal prism*, Fig. 222, is a twelve-sided prism bounded by twelve faces, each one of which is parallel to the vertical axis, and also meets two adjacent horizontal axes at unequal distances, the ratio of which always lies between 1 : 1 and 1 : 2. This prism has two unlike edges, lettered x and y, as shown in Fig. 222. The general symbol is $(hk\bar{i}0)$ and the indices of the faces of a given form, as $(21\bar{3}0)$, are:

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* Since $1a_1 : 1a_2 : -\frac{1}{2}a_3 : \infty c$ is equivalent to $2a_1 : 2a_2 : -1a_3 : \infty c$.

$21\bar{3}0,\quad 1230,\quad \bar{1}3\bar{2}0,\quad 23\bar{1}0,\quad \bar{3}210,\quad \bar{3}120,$
$\bar{2}\bar{1}30,\quad \bar{1}\bar{2}30,\quad 1\bar{3}20,\quad 2\bar{3}10,\quad 3\bar{2}\bar{1}0,\quad 3\bar{1}\bar{2}0.$

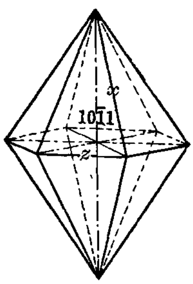
128. Pyramids of the First Order. — Corresponding to the three types of prisms just mentioned, there are three types of pyramids

A *pyramid of the first order*, Fig. 224, is a double six-sided pyramid (or bipyramid) bounded by twelve similar triangular faces — six above and six below — which have the same position relative to the horizontal axes as the faces of the first order prism, while they also intersect the vertical axis above and below. The general symbol is hence $(h0\bar{h}l)$. The faces of a given form, as $10\bar{1}1$), are:

Above $10\bar{1}1,\quad 01\bar{1}1,\quad \bar{1}101,\quad \bar{1}011,\quad 0\bar{1}11,\quad 1\bar{1}01.$
Below $10\bar{1}\bar{1},\quad 01\bar{1}\bar{1},\quad \bar{1}10\bar{1},\quad \bar{1}01\bar{1},\quad 0\bar{1}1\bar{1},\quad 1\bar{1}0\bar{1}.$

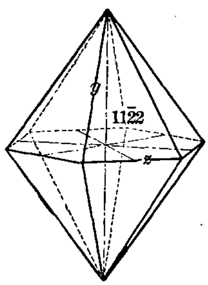
On a given species there may be a number of pyramids of the first order, differing in the ratio of the intercepts on the horizontal to the vertical axis, and thus forming a zone between the base (0001) and the faces of the unit prism $(10\bar{1}0)$. Their symbols, passing from the base (0001) to the unit prism $(10\bar{1}0)$, would be, for example, $10\bar{1}4$, $10\bar{1}2$, $20\bar{2}3$, $10\bar{1}1$, $30\bar{3}2$, $20\bar{2}1$, etc. In Fig. 228 the faces p and u are first order pyramids and they have the symbols respectively $(10\bar{1}1)$ and $(20\bar{2}1)$, here $c = 0.4989$. As shown in these cases the faces of the first order pyramids replace the edges of the first order prism. On the other hand, they replace the solid angles of the second order prism $a(11\bar{2}0)$.

224



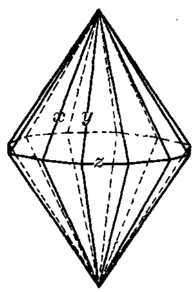
First Order Pyramid

225



Second Order Pyramid

226



Dihexagonal Pyramid

129. Pyramids of the Second Order. — The *pyramid of the second order* (Fig. 225), is a double six-sided pyramid including the twelve similar faces which have the same position relative to the horizontal axes as the faces of the second order prism, and which also intersect the vertical axis. They have the general symbol $(h \cdot h \cdot 2\bar{h} \cdot l)$. The indices of the faces of the form $(11\bar{2}2)$ are:

Above $11\bar{2}2,\quad \bar{1}2\bar{1}2,\quad \bar{2}112,\quad \bar{1}\bar{1}22,\quad 1\bar{2}12,\quad 2\bar{1}\bar{1}2.$
Below $11\bar{2}\bar{2},\quad \bar{1}2\bar{1}\bar{2},\quad \bar{2}11\bar{2},\quad \bar{1}\bar{1}2\bar{2},\quad 1\bar{2}1\bar{2},\quad 2\bar{1}\bar{1}\bar{2}.$

The faces of the second order pyramid replace the edges between the faces of the second order prism and the base. Further, they replace the solid angles of the first order prism $m(10\bar{1}0)$. There may be on a single crystal a number of second order pyramids forming a zone between the base $c(0001)$ and the faces of the second order prism $a(11\bar{2}0)$, as, naming them in order: $11\bar{2}4$, $11\bar{2}2$, $22\bar{4}3$, $11\bar{2}1$, etc. In Fig. 227, s is the second order pyramid $(11\bar{2}1)$.

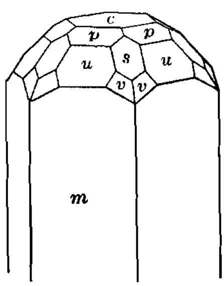
130. Dihexagonal Pyramid. — The *dihexagonal pyramid*, Fig. 226, is a double twelve-sided pyramid, having the twenty-four similar faces embraced under the general symbol $(hk\bar{i}l)$. It is bounded by twenty-four similar faces, each meeting the vertical axis, and also meeting two adjacent horizontal axes at unequal distances, the ratio of which always lies between 1 : 1 and 1 : 2. Thus the form $(21\bar{3}1)$ includes the following twelve faces in the upper half of the crystal:

$$21\bar{3}1,\quad 12\bar{3}1,\quad \bar{1}3\bar{2}1,\quad \bar{2}3\bar{1}1,\quad \bar{3}211,\quad \bar{3}121,$$
$$\bar{2}\bar{1}31,\quad \bar{1}\bar{2}31,\quad 1\bar{3}21,\quad 2\bar{3}11,\quad 3\bar{2}\bar{1}1,\quad 3\bar{1}\bar{2}1.$$

And similarly below with l (here 1) negative, $21\bar{3}\bar{1}$, etc. The dihexagonal pyramid is often called a *berylloid* because a common form with the species beryl. The dihexagonal pyramid $v(21\bar{3}1)$ is shown on Figs. 224, 225.

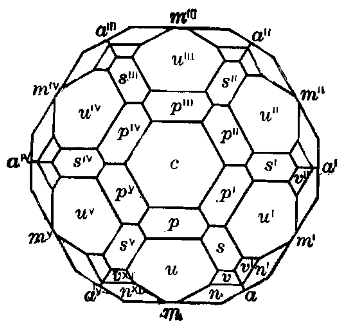
131. Combinations. — Fig. 227 of beryl shows a combination of the

227



Beryl

228



Beryl

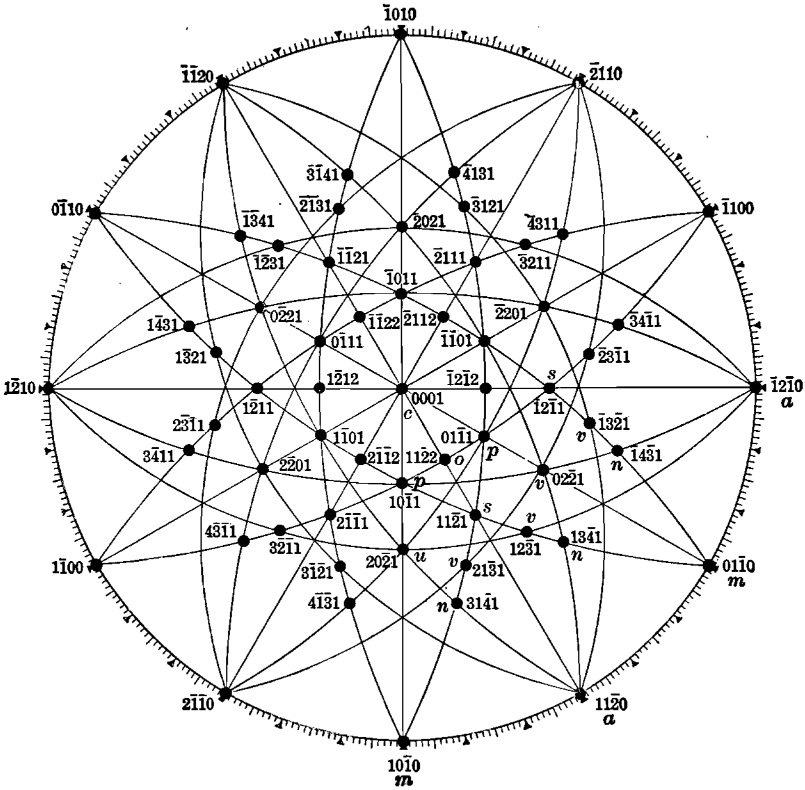
base $c(0001)$ and prism $m(10\bar{1}0)$ with the first order pyramids $p(10\bar{1}1)$ and $u(20\bar{2}1)$; the second order pyramid $s(11\bar{2}1)$ and the dihexagonal pyramid $v(21\bar{3}1)$. Both the last forms lie in a zone between m and s, for which it is true that $k = l$. The basal projection of a similar crystal shown in Fig. 228 is very instructive as exhibiting the symmetry of the normal hexagonal class. This is also true of the stereographic and gnomonic projections in Figs. 229 and 230 of a like crystal with the added form $o(11\bar{2}2)$.

2. HEMIMORPHIC CLASS (14). ZINCITE TYPE

(Dihexagonal Pyramidal or Holohedral Hemimorphic Class)

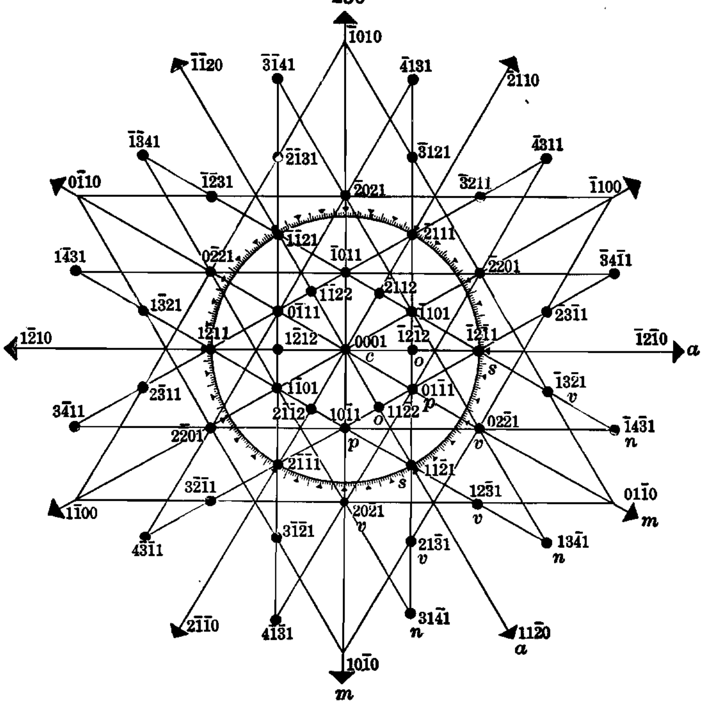
132. Symmetry. — This class differs from the normal class only in having no horizontal plane of principal symmetry and no horizontal axes of binary symmetry. It has, however, the same six vertical planes of symmetry meeting at angles of 30° in the vertical crystallographic axis which is

229



Beryl

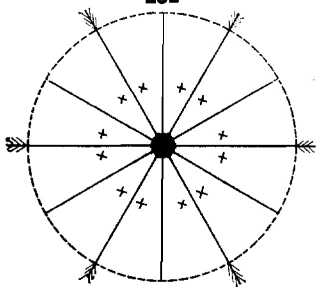
230



Beryl

an axis of hexagonal symmetry. There is no center of symmetry. The symmetry is exhibited in the stereographic projection, Fig. 231.

231



Symmetry of Hemimorphic Class

133. Forms. — The forms belonging to this class are the two basal planes, 0001 and 000$\bar{1}$, here distinct forms, the positive (upper) and negative (lower) pyramids of each of the three types; also the three prisms, which last do not differ geometrically from the prisms of the normal class. An example of this class is found in zincite, Fig. 44, p. 22. Iodyrite, greenockite and wurtzite are also classed here.

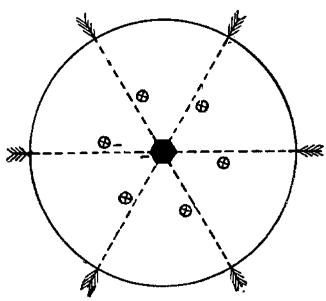
3. TRIPYRAMIDAL CLASS (15). APATITE TYPE

(Hexagonal Bipyramidal or Pyramidal Hemihedral Class)

134. Typical Forms and Symmetry. — This class is important because it includes the common species of the Apatite Group, apatite, pyromorphite, mimetite, vanadinite. The typical form is the hexagonal prism ($hk\bar{i}0$) and the hexagonal pyramid ($hk\bar{i}l$), each designated as of the *third order*. These forms which are shown in Figs. 233 and 234 may be considered as derived from the corresponding dihexagonal forms of the normal class by the omission of one half of the faces of the latter. They and the other forms of the class have only one plane of symmetry, the plane of the horizontal axes, and also one axis of hexagonal symmetry (the vertical axis).

The symmetry is exhibited in the stereographic projection (Fig. 232). It is seen here, as in the figures of crystals given, that, like the tripyramidal class under the tetragonal system, the faces of the general form ($hk\bar{i}l$) present are half of the possible planes belonging to each sectant, and further that those above and below fall in the same vertical zone.

232



Symmetry of Tripyramidal Class

135. Prism and Pyramid of the Third Order. — The prism of the third order (Fig. 233) has six like faces embraced under the general symbol ($hk\bar{i}0$), and the form is a regular hexagonal prism with angles of 60°, not to be distinguished geometrically, if alone, from the other hexagonal prisms; cf. Figs. 220, 221, p. 96. The six faces of the right-handed form (21$\bar{3}$0) have the indices

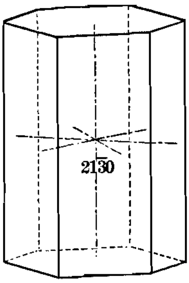
$$21\bar{3}0,\quad \bar{1}3\bar{2}0,\quad \bar{3}210,\quad \bar{2}\bar{1}30,\quad 1\bar{3}20,\quad 3\bar{2}\bar{1}0.$$

The faces of the complementary left-handed form have the indices:

$$12\bar{3}0,\quad \bar{2}3\bar{1}0,\quad \bar{3}120,\quad \bar{1}\bar{2}30,\quad 2\bar{3}10,\quad 3\bar{1}\bar{2}0.$$

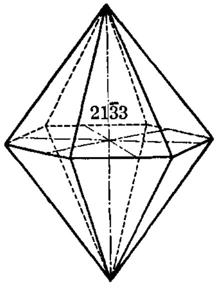
As already stated these two forms together embrace all the faces of the dihexagonal prism (Fig. 222).

233



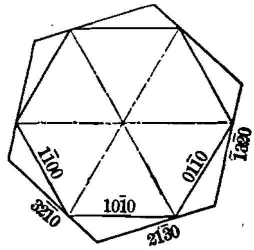
Third Order Prism

234



Third Order Pyramid

235



The pyramid is also a regular double hexagonal pyramid of the third order, and in its relations to the other hexagonal pyramids of the class (Figs. 224, 225) it is analogous to the square pyramid of the third order met with in the corresponding class of the tetragonal system (see Art. **100**). The faces of the right-handed form $(21\bar{3}1)$ are:

Above $21\bar{3}1$, $\bar{1}3\bar{2}1$, $\bar{3}211$, $\bar{2}\bar{1}31$, $1\bar{3}21$, $3\bar{2}\bar{1}1$.
Below $21\bar{3}\bar{1}$, $\bar{1}3\bar{2}\bar{1}$, $\bar{3}21\bar{1}$, $\bar{2}\bar{1}3\bar{1}$, $1\bar{3}2\bar{1}$, $3\bar{2}\bar{1}\bar{1}$.

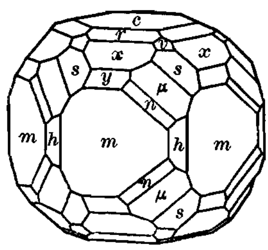
There is also a complementary left-handed form, which with this embraces all the faces of the dihexagonal pyramid. The cross section of Fig. 235 shows in outline the position of the first order prism, and also that of the right-handed prism of the third order.

The prism and pyramid just described do not often appear on crystals as predominating forms, though this is sometimes the case, but commonly these faces are present modifying other fundamental forms.

136. Other Forms. — The remaining forms of the class are geometrically like those of the normal class, viz., the base (0001); the first order prism $(10\bar{1}0)$; the second order prism $(11\bar{2}0)$; the first order pyramids $(h0\bar{h}l)$; and the second order pyramids $(h\cdot h\cdot 2\bar{h}\cdot l)$. That their molecular structure, however, corresponds to the symmetry of this class is readily proved, for example, by etching. In this way it was shown that pyromorphite and mimetite belonged in the same group with apatite (Baumhauer), though crystals with the typical forms had not been observed. This class is given its name of *Tripyramidal* because its forms include three distinct types of pyramids.

137. A typical crystal of apatite is given in Fig. 236. It shows the third order prism $h(21\bar{3}0)$, and the third order pyramids, $\mu(21\bar{3}1)$, $n(31\bar{4}1)$; also the first order pyramids $r(10\bar{1}2)$, $x(10\bar{1}1)$, $y(20\bar{2}1)$, the second order pyramids $v(11\bar{2}2)$, $s(11\bar{2}1)$; finally, the prism $m(10\bar{1}0)$, and the base $c(0001)$.

236



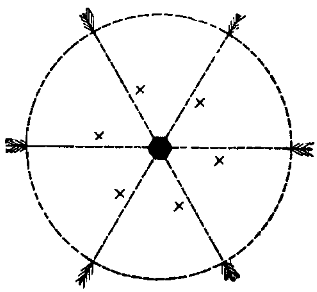
Apatite

4. PYRAMIDAL-HEMIMORPHIC CLASS (16). NEPHELITE TYPE

(Hexagonal Pyramidal or Pyramidal Hemihedral Hemimorphic Class)

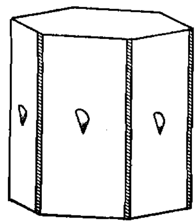
138. Symmetry. — A fourth class under the hexagonal division, the *pyramidal-hemimorphic class*, is like that just described, except that the forms are hemimorphic. The single horizontal plane of symmetry is absent, but the vertical axis is still an axis of hexagonal symmetry. This symmetry is shown in the stereographic projection of Fig. 237. The typical form would be like the upper half of Fig. 234 of the pyramid of the third order. The species nephelite is shown by the character of the etching-figures (Fig. 238, Groth after Baumhauer) to belong here.

237



Symmetry of Pyramidal-Hemimorphic Class

238



Nephelite

5. TRAPEZOHEDRAL CLASS (17)

(*Hexagonal Trapezohedral or Trapezohedral Hemihedral Class*)

139. Symmetry. — The last class of this division is the *trapezohedral class*. It has no plane of symmetry, but the vertical axis is an axis of hexagonal symmetry, and there are, further, six horizontal axes of binary symmetry. There is no center of symmetry. The symmetry and the distribution of the faces of the typical form ($hk\bar{i}l$) is shown in the stereographic projection (Fig. 239). The typical forms may be derived from the dihexagonal pyramid by the omission of the alternate faces of the latter. There are two possible types known as the right and left hexagonal trapezohedrons (see

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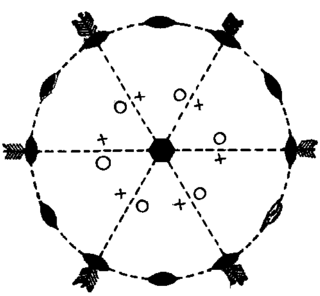
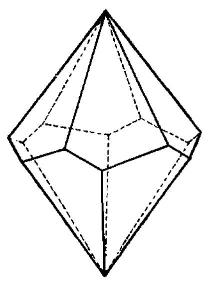


Symmetry of Trapezohedral Class

240



Hexagonal Trapezohedron

Fig. 240), which are enantiomorphous, and the few crystallized salts falling in this class show circular polarization. A modification of quartz known as β-quartz is also described as belonging here. The indices of the right form $(21\bar{3}1)$ are as follows:

Above	$21\bar{3}1$,	$\bar{1}3\bar{2}1$,	$\bar{3}211$,	$\bar{2}\bar{1}31$,	$1\bar{3}21$,	$3\bar{2}\bar{1}1$.
Below	$12\bar{3}\bar{1}$,	$\bar{2}3\bar{1}\bar{1}$,	$\bar{3}12\bar{1}$,	$\bar{1}\bar{2}3\bar{1}$,	$2\bar{3}1\bar{1}$,	$3\bar{1}\bar{2}\bar{1}$.

B. Trigonal or Rhombohedral Division

(Trigonal System)

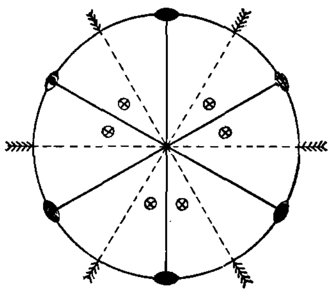
140. *General Character.* — As stated on p. 19, the classes of this division are characterized by a vertical axis of trigonal, or threefold, symmetry. There are seven classes here included of which the rhombohedral class of the Calcite Type is by far the most important.

1. TRIGONAL CLASS (18). BENITOITE TYPE

(Ditrigonal Bipyramidal, Trigonal Hemihedral or Trigonotype Class)

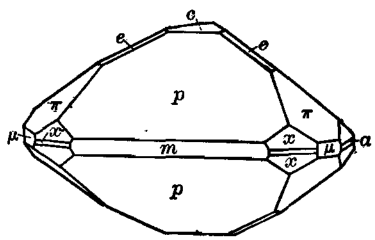
141. Typical Forms and Symmetry. — This class has, besides the vertical axis of trigonal symmetry, three horizontal axes of binary symmetry which are diagonal to the crystallographic axes. There are four planes of symmetry, one horizontal, and three vertical diagonal planes intersecting at angles of 60° in the vertical axis. The symmetry and the distribution of the faces of the positive ditrigonal pyramid is shown in Fig. 241. The characteristic forms are as follows. *Trigonal prism* consisting of three faces comprising one half the faces of the hexagonal prism of the first order. They are of two types, called positive $(10\bar{1}0)$ and negative $(01\bar{1}0)$. *Trigonal*

241



Symmetry of Trigonal Class

242



Benitoite (Palache)

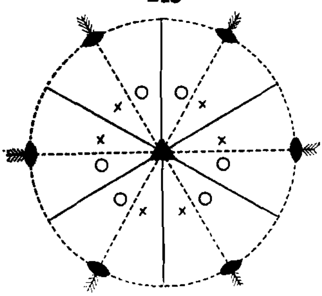
pyramid, a double three-faced pyramid, consisting of six faces corresponding to one half the faces of the hexagonal pyramid of the first order. The faces of the upper and lower halves fall in vertical zones with each other. There are two types, called positive $(10\bar{1}1)$ and negative $(01\bar{1}1)$. *Ditrigonal prism* consists of six vertical faces arranged in three similar sets of two faces and having therefore the alternate edges of differing character. It may be derived from the dihexagonal prism by taking alternating pairs of faces. *Ditrigonal pyramid* consists of twelve faces, six above and six below. It, like the prism, may be derived from the dihexagonal form by taking alternate pairs of faces of the latter. The faces of the upper and lower halves fall in vertical zones. The only representative of this class known is the rare mineral benitoite, a crystal of which is represented in Fig. 242. This crystal shows the trigonal prisms $m(10\bar{1}0)$ and $\mu(01\bar{1}0)$, the hexagonal prism of the second order, $a(11\bar{2}0)$, the trigonal pyramids, $p(10\bar{1}1)$ and $\pi(01\bar{1}1)$; $e(01\bar{1}2)$ and the hexagonal pyramid of the second order, $x(22\bar{4}1)$

2. RHOMBOHEDRAL CLASS (19). CALCITE TYPE

(Ditrigonal Scalenohedral or Rhombohedral Hemihedral Class)

142. Typical Forms and Symmetry. — The typical forms of the *rhombohedral class* are the rhombohedron (Fig. 244) and the scalenohedron (Fig. 259). These forms, with the projections, Figs. 243 and 269, illustrate the symmetry characteristic of the class. There are three planes of symmetry only; these are diangoal to the horizontal crystallographic axes and intersect at angles of 60° in the vertical crystallographic axis. This axis is with these forms an axis of trigonal symmetry; there are, further, three horizontal axes diagonal to the crystallographic axes of binary symmetry. Compare Fig. 244, also Fig. 245 *et seq.*

243

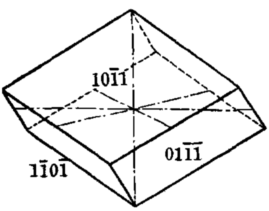


Symmetry of Rhombohedral Class

By comparing Fig. 269 with Fig. 229, p. 99, it will be seen that all the faces in half the sectants are present. This group is hence analogous to the tetrahedral class of the isometric system, and the sphenoidal class of the tetragonal system.

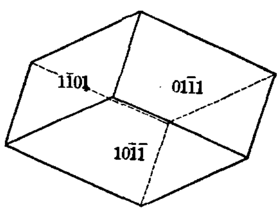
143. Rhombohedron. — Geometrically described, the *rhombohedron* is a solid bounded by six like faces, each a rhomb. It has six like lateral edges forming a zigzag line about the crystal, and six like terminal edges, three above and three in alternate position below. The vertical axis joins the two trihedral solid angles, and the horizontal axes join the middle points of the opposite sides, as shown in Fig. 244.

244



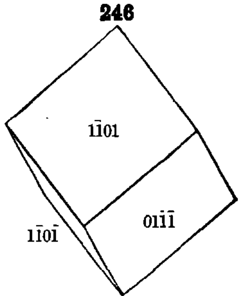
Positive Rhombohedron Calcite

245



Negative Rhombohedron

246



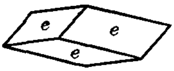
Positive Rhombohedron Hematite

The general symbol of the rhombohedron is $(h0\bar{h}l)$, and the successive faces of the unit form $(10\bar{1}1)$ have the indices:

Above, $10\bar{1}1$, $\bar{1}101$, $0\bar{1}11$; below, $01\bar{1}\bar{1}$, $\bar{1}01\bar{1}$, $1\bar{1}0\bar{1}$.

The geometrical shape of the rhombohedron varies widely as the angles change, and consequently the relative length of the vertical axis c (expressed in terms of the horizontal axes, $a = 1$). As the vertical axis diminishes, the rhombohedrons become more and more obtuse or flattened; and as it increases they become more and more acute. A cube placed with an octahedral axis vertical is obviously the limiting case between the obtuse and acute forms where the interfacial angle is 90°. In Fig. 244 of calcite the normal rhombohedral angle is 74° 55′ and $c = 0{\cdot}854$, while for Fig. 246 of hematite this angle is 94° and $c = 1{\cdot}366$. Further, Figs. 246–251 show other rhombohedrons of calcite, namely, $l\,(01\bar{1}2)$, $\phi\,(05\bar{5}4)$, $f(02\bar{2}1)$, $M(40\bar{4}1)$, and $\rho(16{\cdot}0{\cdot}\overline{16}{\cdot}1)$; here the vertical axes are in the ratio of $\frac{1}{2}$, $\frac{5}{4}$, 2, 4, 16, to that of the fundamental (cleavage) rhombohedron of Fig. 244, whose angle determines the value of c.

247



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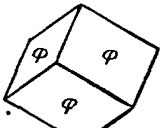

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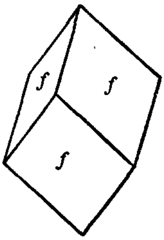

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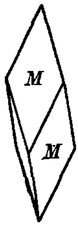

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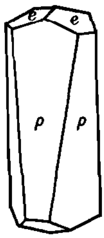

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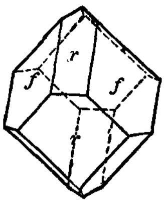

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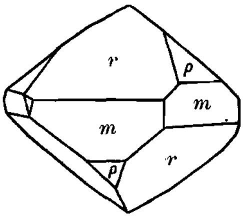
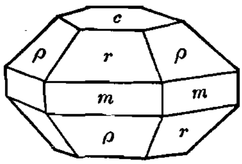


254



Figs. 247-252, Calcite Figs. 253-254, Gmelinite

144. Positive and Negative Rhombohedrons. — To every positive rhombohedron there may be an inverse and complementary form, identical geometrically, but bounded by faces falling in the alternate sectants. Thus the negative form of the unit rhombohedron $(01\bar{1}1)$ shown in Fig. 245 has the faces:

Above, $01\bar{1}1$, $\bar{1}011$, $1\bar{1}01$; below, $\bar{1}10\bar{1}$, $0\bar{1}1\bar{1}$, $10\bar{1}\bar{1}$.

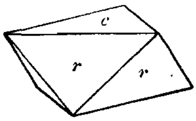
The position of these in the projections (Figs. 269, 270) should be carefully studied. Of the figures already referred to, Figs. 244, 246, 250 are positive, and Figs. 245, 247, 248, 249 negative, rhombohedrons; Fig. 251 shows both forms.

It will be seen that the two complementary positive and negative rhombohedrons of given axial length together embrace all the like faces of the double six-sided hexagonal pyramid of the first order. When these two rhombohedrons are equally developed the form is geometrically identical with this pyramid. This is illustrated by Fig. 254 of gmelinite $r(10\bar{1}1)$,

$\rho(01\bar{1}1)$ and by Figs. 284, 285, p. 113, of quartz, $r(10\bar{1}1)$, $z(01\bar{1}1)$.* In each case the form, which is geometrically a double hexagonal pyramid (in Fig. 254 with c and m), is in fact a combination of the two unit rhombohedrons, positive and negative. Commonly a difference in size between the two forms may be observed, as in Figs. 253 and 286, where the form taken as the positive rhombohedron predominates. But even if this distinction cannot be established, the two rhombohedrons can always be distinguished by etching, or, as in the case of quartz, by pyro-electrical phenomena.

145. Of the two series, or zones, of rhombohedrons the faces of the *positive rhombohedrons* replace the edges between the base (0001) and the first order prism $(10\bar{1}0)$. Also the faces of the *negative rhombohedrons* replace the alternate edges of the same forms, that is, the edges between (0001) and $(01\bar{1}0)$ (compare Figs. 253, 254, etc.). Fig. 255 shows the rhombohedron in combination with the base. Fig. 256 the same with the prism $a(11\bar{2}0)$. When the angle between the two forms happens to approximate to 70° 32′ the crystal simulates the aspect of a regular octahedron. This is illustrated by Fig. 257; here co = 69° 42′, also oo = 71° 22′, and the crystal resembles closely an octahedron with truncated edges (cf. Fig. 99, p. 55).

255



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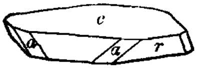

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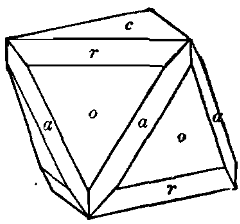

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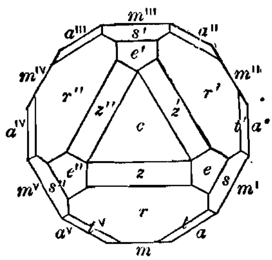


Figs. 255, 256, Hematite Coquimbite Eudialyte

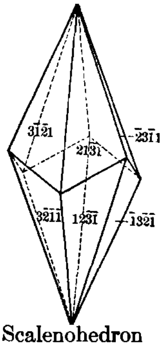
146. There is a very simple relation between the positive and negative rhombohedrons which it is important to remember. The form of one series which truncates the terminal edges of a given form of the other will have one half the intercept on the vertical crystallographic axis of the latter. This ratio is expressed in the values of the indices of the two forms. Thus $(01\bar{1}2)$, truncates the terminal edges of the positive unit rhombohedron $(10\bar{1}1)$; $(10\bar{1}4)$ truncates the terminal edges of $(01\bar{1}2)$, $(10\bar{1}5)$ of $(20\bar{2}5)$. Again $(10\bar{1}1)$ truncates the edges of $(02\bar{2}1)$, $(40\bar{4}1)$ of $(02\bar{2}1)$, etc. This is illustrated by Fig. 252 with the forms $r(10\bar{1}1)$ and $f(02\bar{2}1)$. Also in Fig. 258, a basal projection, $z(10\bar{1}4)$ truncates the edges of $e(01\bar{1}2)$; $e(01\bar{1}2)$ of $r(10\bar{1}1)$; $r(10\bar{1}1)$ of $s(02\bar{2}1)$.

147. Scalenohedron. — The *scalenohedron*, shown in Fig. 259, is the general form for this class corresponding to the symbol $hk\bar{i}l$. It is a solid, bounded by twelve faces, each a scalene triangle. It has roughly the shape of a double six-sided pyramid, but there are two sets of terminal edges, one more obtuse than the other, and the lateral edges form a zigzag edge around the form like that of the rhombohedron. It may be considered as derived from the dihexagonal pyramid by taking the alternating pairs of faces of

* Quartz serves as a convenient illustration in this case, none the less so notwithstanding the fact that it belongs to the trapezohedral class of this division.

that form. It is to be noted that the faces in the lower half of the form do not fall in vertical zones with those of the upper half. Like the rhombohedrons, the scalenohedrons may be either positive or negative. The positive forms correspond in position to the positive rhombohedrons and conversely.

259



Scalenohedron

The positive scalenohedron $(21\bar{3}1)$, Fig. 259, has the following indices for the several faces:

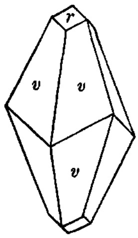
Above	$21\bar{3}1$,	$\bar{2}3\bar{1}1$,	$\bar{3}211$,	$\bar{1}\bar{2}31$,	$1\bar{3}21$,	$3\bar{1}\bar{2}1$
Below	$12\bar{3}\bar{1}$,	$\bar{1}3\bar{2}\bar{1}$,	$\bar{3}12\bar{1}$,	$\bar{2}\bar{1}3\bar{1}$,	$2\bar{3}1\bar{1}$,	$3\bar{2}\bar{1}\bar{1}$.

For the complementary negative scalenohedron $(12\bar{3}1)$ the indices of the faces are:

Above	$12\bar{3}1$,	$\bar{1}3\bar{2}1$,	$\bar{3}121$,	$\bar{2}\bar{1}31$,	$2\bar{3}11$,	$3\bar{2}\bar{1}1$.
Below	$\bar{2}3\bar{1}\bar{1}$,	$\bar{3}21\bar{1}$,	$\bar{1}\bar{2}3\bar{1}$,	$1\bar{3}2\bar{1}$,	$3\bar{1}\bar{2}\bar{1}$,	$21\bar{3}\bar{1}$.

148. Relation of Scalenohedrons to Rhombohedrons. — It was noted above that the scalenohedron in general has a series of zigzag lateral edges like the rhombohedron. It is obvious, further, that for every rhombohedron there will be a series or zone of scalenohedrons having the *same* lateral edges. This is shown in Fig. 262, where the scalenohedron

260.



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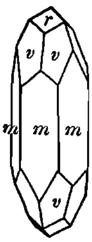

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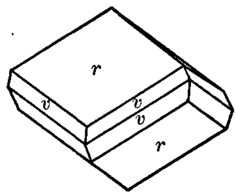

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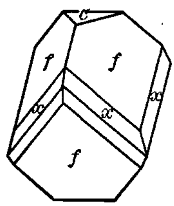
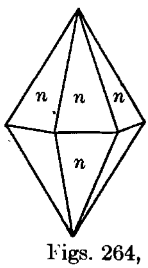


Figs. 260-263, Calcite

264



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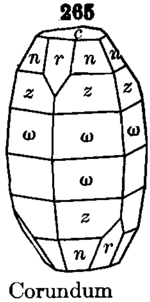
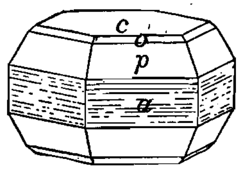
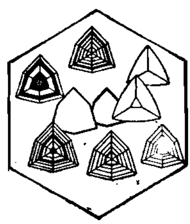


Figs. 264, 265, Corundum

266



267



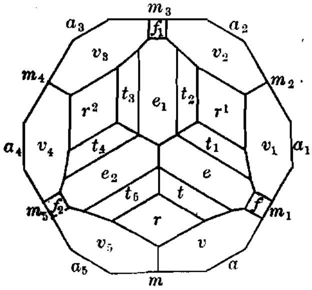
Figs. 266, 267, Spangolite*

$v(21\bar{3}1)$ bevels the lateral edges of the fundamental rhombohedron $r(10\bar{1}1)$; the same would be true of the scalenohedron $(32\bar{5}1)$, etc. Further, in Fig. 263, the negative scalenohedron $x(13\bar{4}1)$ bevels the lateral edges of the negative rhombohedron $f(02\bar{2}1)$. The relation of the indices which must exist in these cases may be shown to be, for example, for the rhombohedron $r(10\bar{1}1)$, $h = k + l$; again for $f(02\bar{2}1)$, $h + 2l = k$, etc. See also the projections, Figs. 269, 270. Further, the position of the scalenohedron may be defined with reference to its parent rhombohedron. For example, in Fig. 262 the scalenohedron $v(21\bar{3}1)$ has three times the vertical axis of the unit rhombohedron $r(10\bar{1}1)$. Again in Fig. 263 $x(13\bar{4}1)$ has twice the vertical axis of $f(02\bar{2}1)$.

* Spangolite belongs properly to the next (hemimorphic) group, but this fact does not destroy the value of the illustration.

149. Other Forms. — The remaining forms of the normal class of the rhombohedral division are geometrically like those of the corresponding class of the hexagonal division — viz., the base $c(0001)$; the prisms $m(10\bar{1}0)$, $a(11\bar{2}0)$, $(hk\bar{i}0)$; also the second order pyramids, as $(11\bar{2}1)$. Some of these forms are shown in the accompanying figures. For further illustrations reference may be made to typical rhombohedral species, as calcite, hematite, etc.

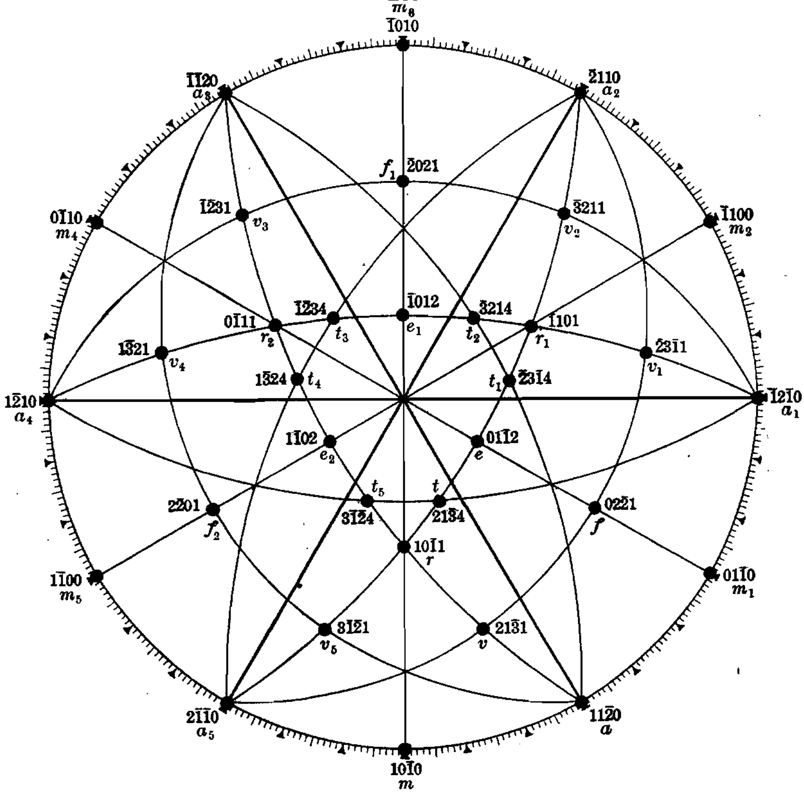
268



Calcite

With respect to the second order pyramid, it is interesting to note that if it occurs alone (as in Fig. 264, $n = 22\bar{4}3$) it is impossible to say, on geometrical grounds, whether it has the trigonal symmetry of the rhombohedral type or the hexagonal symmetry of the hexagonal type. In the latter case,

269

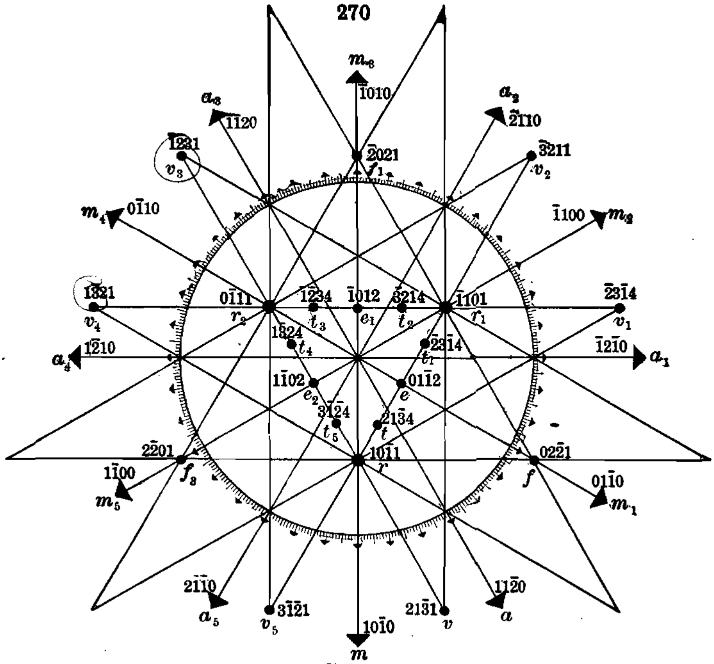


Calcite

the form might be made a first order pyramid by exchanging the axial and diagonal planes of symmetry. The true symmetry, however, is often indicated, as with corundum, by the occurrence on other crystals of rhombohedral faces, as $r(10\bar{1}1)$ in Fig. 265 (here $z = 22\bar{4}1$, $\omega = 14{\cdot}14{\cdot}\overline{28}{\cdot}3$). Even if rhombohedral faces are absent (Fig. 266), the etching-figures (Fig. 267) will often serve to reveal the true trigonal molecular symmetry; here $o = (11\bar{2}4)$, $p = (11\bar{2}2)$.

150. A basal projection of a somewhat complex crystal of calcite is given in Fig. 268, and stereographic and gnomonic projections of the same forms in Figs. 269 and 270; both show well the symmetry in the distribution

270



Calcite

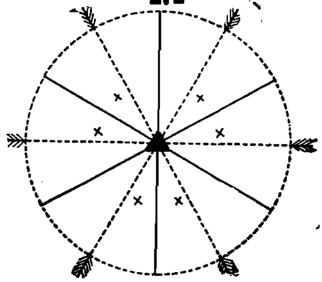
of the faces. Here the forms are: prisms, $a(11\bar{2}0)$, $m(10\bar{1}0)$; rhombohedrons, positive, $r(10\bar{1}1)$, negative, $e(01\bar{1}2)$, $f(02\bar{2}1)$; scalenohedrons, positive, $v(21\bar{3}1)$, $t(21\bar{3}4)$.

3. RHOMBOHEDRAL-HEMIMORPHIC CLASS (20). TOURMALINE TYPE

(Ditrigonal Pyramidal or Trigonal Hemihedral Hemimorphic Class)

151. Symmetry. — A number of prominent rhombohedral species, as tourmaline, pyrargyrite, proustite, belong to a hemimorphic class under this division. For them the symmetry in the grouping of the faces differs at the two extremities of the vertical axis. The forms have the same three diagonal planes of symmetry meeting at angles of 60° in the vertical axis,

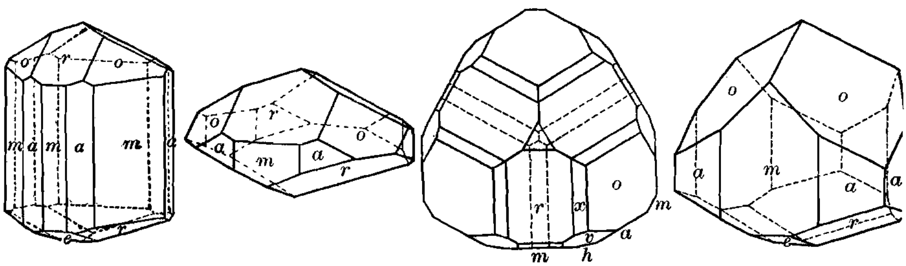
271



Symmetry of Rhombohedral-Hemimorphic Class which is an axis of trigonal symmetry. There are, however, no horizontal axes of symmetry, as in the rhombohedral class, and there is no center of symmetry. Cf. Fig. 271.

152. Typical Forms. — In this class the basal planes (0001) and $(000\bar{1})$ are distinct forms. The other characteristic forms are the two trigonal prisms $m(10\bar{1}0)$ and $m_{,}(01\bar{1}0)$ of the first order series; also the four trigonal first order pyramids, corresponding respectively to the three upper and three lower faces of a positive rhombohedron, and the three upper and three lower faces of the negative rhombohedron; also the hemimorphic second order hexagonal pyramid; finally, the four ditrigonal pyramids, corresponding to the upper and lower faces respectively of the positive and negative scalenohedrons. Figs. 272–275 illustrate these forms. Fig. 274 is a basal section with $r_{,}(01\bar{1}\bar{1})$ and $e_{,}(10\bar{1}\bar{2})$ below.

272 273 274 275



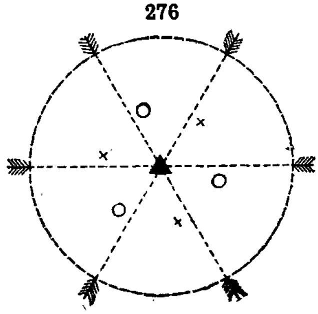
Figs. 272-275, Tourmaline

4. TRI-RHOMBOHEDRAL CLASS (21). PHENACITE TYPE

(Rhombohedral or Rhombohedral Tetartohedral Class)

153. Symmetry. — This class, illustrated by the species dioptase, phenacite, willemite, dolomite, ilmenite, etc., is an important one. It is characterized by the absence of all planes of symmetry, but the vertical axis is still an axis of trigonal symmetry, and there is a center of symmetry. Cf. Fig. 276.

276



Symmetry of Tri-Rhombohedral Class

154. Typical Forms. — The distinctive forms of the class are the rhombohedron of the second order and the hexagonal prism and rhombohedron, each of the third order. The class is thus characterized by three rhombohedrons of distinct types (each + and −), and hence the name given to it.

The *second order rhombohedron* may be derived by taking one half the faces of the normal hexagonal pyramid of the second order. There will be two complementary forms known as positive and negative. For example, in a given case the indices of the faces for the positive and negative forms are:

Positive	(above)	$11\bar{2}2$,	$\bar{2}112$,	$1\bar{2}12$;	(below)	$\bar{1}2\bar{1}\bar{2}$,	$\bar{1}\bar{1}2\bar{2}$, $2\bar{1}\bar{1}\bar{2}$,
Negative	(above)	$\bar{1}2\bar{1}2$,	$\bar{1}\bar{1}22$,	$2\bar{1}\bar{1}2$;	(below)	$\bar{2}11\bar{2}$,	$1\bar{2}1\bar{2}$, $11\bar{2}\bar{2}$.

The *rhombohedron of the third order* has the general symbol $(hk\bar{i}l)$, and may be derived from the normal dihexagonal pyramid, Fig. 226, by taking one quarter of the faces of the latter.

There are therefore four complementary third order rhombohedrons, distinguished respectively as positive right-handed $(21\bar{3}1)$, positive left-handed $(3\bar{1}\bar{2}1)$, negative right-handed $(\bar{1}3\bar{2}1)$, and negative left-handed $(12\bar{3}1)$. The indices of the six like faces of the positive right-handed form $(21\bar{3}1)$ are:

Above $21\bar{3}1$, $\bar{3}211$, $1\bar{3}21$; below $\bar{1}3\bar{2}\bar{1}$, $\bar{2}\bar{1}3\bar{1}$, $3\bar{2}\bar{1}\bar{1}$.

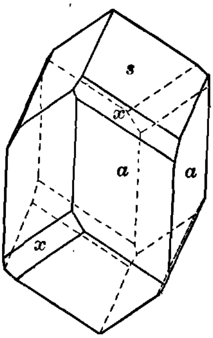
The *hexagonal prism of the third order* may be derived from the normal dihexagonal prism, Fig. 219, by taking one half the faces of the latter. There are two complementary forms known as right- and left-handed. The faces of these forms in a given case $(21\bar{3}0)$ have the indices:

Right	$21\bar{3}0$,	$\bar{1}3\bar{2}0$,	$\bar{3}210$,	$\bar{2}\bar{1}30$,	$1\bar{3}20$,	$3\bar{2}\bar{1}0$,
Left	$12\bar{3}0$	$\bar{2}3\bar{1}0$	$\bar{3}120$	$\bar{1}\bar{2}30$	$2\bar{3}10$,	$3\bar{1}\bar{2}0$.

155. The remaining forms are geometrically like those of the rhombohedral class, viz.: Base $c(0001)$; first order prism $m(10\bar{1}0)$; second order prism $a(11\bar{2}0)$; rhombohedrons of the first order, as $(10\bar{1}1)$ and $(01\bar{1}1)$, etc.

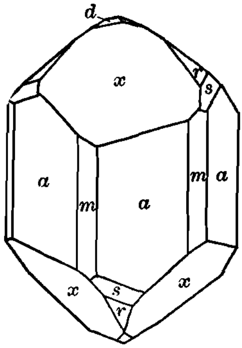
156. The forms of this group are illustrated by Figs. 277–279. Fig. 277 is of dioptase and shows the hexagonal prism of the second order $a(11\bar{2}0)$ with a negative first order rhombohedron, $s(02\bar{2}1)$ and the third order rhombohedron $x(13\bar{4}1)$. Figs. 278 and 279 show the horizontal and clinographic

277



Dioptase

278



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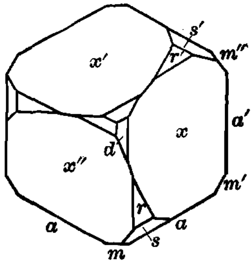

Phernacite

projections of a crystal of phenacite with the following forms: first order prism, $m(10\bar{1}0)$; second order prism, $a(11\bar{2}0)$; third order rhombohedrons, $x(12\bar{3}2)$ and $s(21\bar{3}1)$; first order rhombohedrons, $r(10\bar{1}1)$ and $d(01\bar{1}2)$.

In order to make clearer the relation of the faces of the different types of forms under this class, Fig. 280 is added. Here the zones of the positive and negative rhombohedrons of the first order are indicated ($+R$ and $-R$) also the general positions of the four types of the third order rhombohedrons ($+r$, $-r$, $+l$, $-l$).

The following scheme may also be helpful in connection with Fig. 280. It

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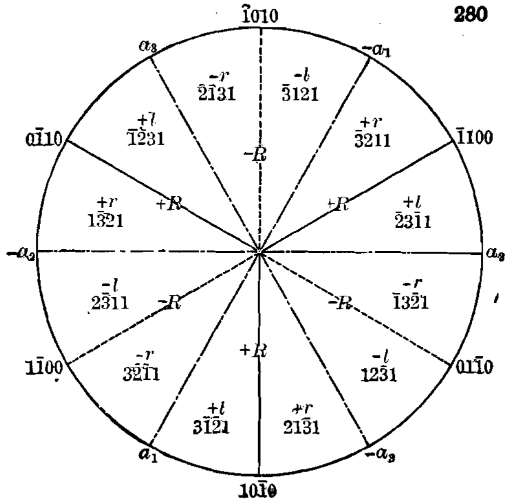


shows the distribution of the faces of the four rhombohedrons of the third order ($+r$, $+l$, $-r$, $-l$) relatively to the faces of the unit hexagonal prism $(10\bar{1}0)$.

Phenacite Type

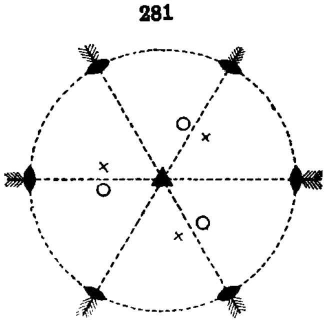
$+l$ $+r$	$-l$ $-r$	$+l$ $+r$	$-l$ $-r$	$+l$ $+r$	$-l$ $-r$
$3\bar{1}\bar{2}1$ $21\bar{3}1$	$12\bar{3}1$ $\bar{1}3\bar{2}1$	$\bar{2}3\bar{1}1$ $\bar{3}211$	$\bar{3}121$ $\bar{2}\bar{1}31$	$\bar{1}\bar{2}31$ $1\bar{3}21$	$2\bar{3}11$ $3\bar{2}\bar{1}1$
$10\bar{1}0$	$01\bar{1}0$	$\bar{1}100$	$\bar{1}010$	$0\bar{1}10$	$1\bar{1}00$
$-l$ $-r$	$+l$ $+r$	$-l$ $-r$	$+l$ $+r$	$-l$ $-r$	$+l$ $+r$
$3\bar{1}\bar{2}\bar{1}$ $21\bar{3}\bar{1}$	$12\bar{3}\bar{1}$ $\bar{1}3\bar{2}\bar{1}$	$\bar{2}3\bar{1}\bar{1}$ $\bar{3}21\bar{1}$	$\bar{3}12\bar{1}$ $\bar{2}\bar{1}3\bar{1}$	$\bar{1}\bar{2}3\bar{1}$ $1\bar{3}2\bar{1}$	$2\bar{3}1\bar{1}$ $3\bar{2}\bar{1}\bar{1}$

5. TRAPEZOHEDRAL CLASS (22). QUARTZ TYPE

(Trigonal Trapezohedral or Trapezohedral Tetartohedral Class)

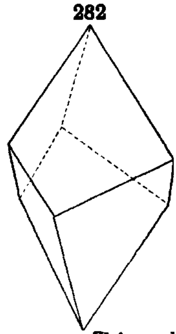
157. Symmetry. — This class includes, among minerals, the species quartz and cinnabar. The forms have no plane of symmetry and no center of symmetry; the vertical axis is, however, an axis of trigonal symmetry, and there are also three horizontal axes of binary symmetry, coinciding in direction with the crystallographic axes; cf. Fig. 281

281

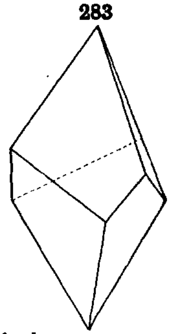


Symmetry of Trapezohedral Class

282



283



Trigonal Trapezohedrons

158. Typical Forms. — The characteristic form of the class is the trigonal trapezohedron shown in Fig. 282. This is the general form corresponding to the symbol $(hk\bar{i}l)$, the faces being distributed as indicated in the accompanying stereographic projection (Fig. 281). The faces of this form correspond to one quarter of the faces of the normal dihexagonal pyramid, Fig. 226. There are therefore four such trapezohedrons, two positive, called respectively right-handed (Fig. 282) and left-handed (Fig. 283), and two similar negative forms, also right- and left-handed (see the scheme given in Art. **160**). It is obvious that the two forms of Figs. 282, 283 are enantiomorphous, and circular polarization is a striking character of the species belonging to the class as elsewhere discussed.

The indices of the six faces belonging to each of these will be evident on consulting Figs. 281 and 229 and 230. The complementary positive form (r and l) of a given symbol include the twelve faces of a positive scalenohedron, while the faces of all four as already stated include the twenty-four faces of the dihexagonal pyramid.

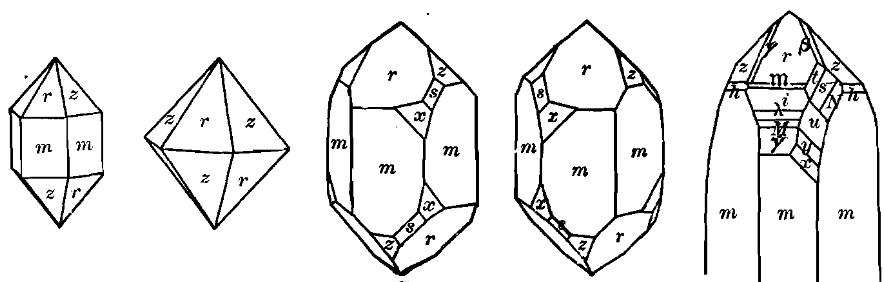
Corresponding to these trapezohedrons there are two *ditrigonal prisms*, respectively right- and left-handed, as $(21\bar{3}0)$ and $(3\bar{1}\bar{2}0)$.

The remaining characteristic forms are the right- and left-handed *trigonal prism* $a(11\bar{2}0)$ and $a(2\bar{1}\bar{1}0)$; also the right- and left-handed *trigonal pyramid*, as $(11\bar{2}2)$ and $(2\bar{1}\bar{1}2)$. They may be derived by taking respectively one half the faces of the hexagonal prism of the second order $(11\bar{2}0)$ or of the corresponding pyramid $(11\bar{2}2)$; these are shown in Figs. 221 and 225.

159. Other Forms. — The other forms of the class are geometrically like those of the normal class. They are the base $c(0001)$, the hexagonal first order prism $m(10\bar{1}0)$, and the positive and negative rhombohedrons as $(10\bar{1}1)$ and $(01\bar{1}1)$. These cannot be distinguished geometrically from the normal forms.

160. Illustrations. — The forms of this class are best shown in the species quartz. As already remarked (p. 106), simple crystals often appear to be of normal hexagonal symmetry, the rhombohedrons $r(10\bar{1}1)$ and $z(01\bar{1}1)$ being equally developed (Figs. 284, 285). In many cases, however, a difference in molecular character between them can be observed, and more com-

284 **285** **286** **287** **288**



Figs. 284-288, Quartz

monly one rhombohedron, $r(10\bar{1}1)$, predominates in size; the distinction can always be made out by etching. Some crystals, like Fig. 286, show as modifying faces the *right* trigonal pyramid $s(11\bar{2}1)$, with a *right positive* trapezohedron, as $x(51\bar{6}1)$. Such crystals are called **right-handed** and **rotate** the plane of polarization of light transmitted in the direction of the vertical axis to the right. A crystal, like Fig. 287, with the *left* trigonal pyramid $s(2\bar{1}11)$ and one or more *left* trapezohedrons, as $x(6\bar{1}\bar{5}1)$, is called left-handed, and as regards light has the opposite character to the crystal of Fig. 286. Fig. 288 shows a more complex right-handed crystal with several positive and negative rhombohedrons, several positive right trapezohedrons and the negative left trapezohedron, N.

The following scheme shows the distribution of the faces of the four trapezohedrons ($+r$, $+l$, $-r$, $-l$) relatively to the faces of the unit hexagonal prism $(10\bar{1}0)$; it is to be compared with the corresponding scheme, given in Art. **156**, of crystals of the phenacite type. In the case of the negative forms some authors prefer to make the faces $21\bar{3}\bar{1}$, $12\bar{3}1$, etc., *right*, and $3\bar{1}\bar{2}\bar{1}$, $\bar{1}3\bar{2}1$, etc., *left*.

Quartz Type

$+l$ $+r$	$-l$ $-r$	$+l$ $+r$	$-l$ $-r$	$+l$ $+r$	$-l$ $-r$
$3\bar{1}\bar{2}1$ $21\bar{3}1$	$12\bar{3}1$ $\bar{1}3\bar{2}1$	$\bar{2}3\bar{1}1$ $\bar{3}211$	$\bar{3}121$ $\bar{2}\bar{1}31$	$\bar{1}\bar{2}31$ $1\bar{3}21$	$2\bar{3}11$ $3\bar{2}\bar{1}1$
$10\bar{1}0$	$01\bar{1}0$	$\bar{1}100$	$\bar{1}010$	$0\bar{1}10$	$1\bar{1}00$
$-r$ $-l$	$+r$ $+l$	$-r$ $-l$	$+r$ $+l$	$-r$ $-l$	$+r$ $+l$
$3\bar{1}\bar{2}\bar{1}$ $21\bar{3}\bar{1}$	$12\bar{3}\bar{1}$ $\bar{1}3\bar{2}\bar{1}$	$\bar{2}3\bar{1}\bar{1}$ $\bar{3}21\bar{1}$	$\bar{3}12\bar{1}$ $\bar{2}\bar{1}3\bar{1}$	$\bar{1}\bar{2}3\bar{1}$ $1\bar{3}2\bar{1}$	$2\bar{3}1\bar{1}$ $3\bar{2}\bar{1}\bar{1}$

161. Other Classes. — The next class (23) is known as the Trigonal Bipyramidal or Trigonal Tetartohedral class. It has one plane of symmetry — that of the horizontal axes, and one axis of trigonal symmetry — the vertical axis. There is no center of symmetry. Its characteristic forms are the three types of trigonal prisms and the three corresponding types of trigonal pyramids. Cf. Fig. 289. This class has no known representation among crystals.

The last class (24) of this division is known as the Trigonal Pyramidal or Trigonal Tetartohedral Hemimorphic class. It has no plane of symmetry

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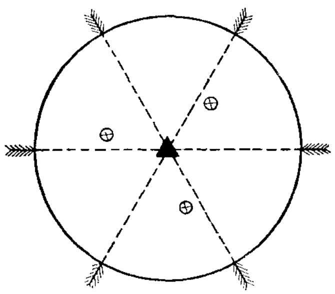
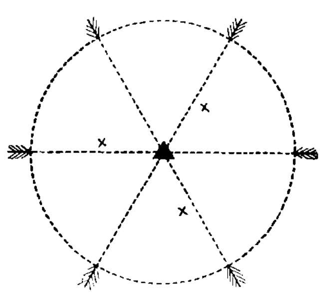


Symmetry of Trigonal Bipyramidal Class

290



Symmetry of the Trigonal Pyramidal Class

and no center of symmetry, but the vertical axis is an axis of trigonal symmetry. The forms are all hemimorphic, the prisms trigonal prisms, and the pyramids hemimorphic trigonal pyramids. Cf. Fig. 290. The crystals of sodium periodate belong to this class.

Mathematical Relations of the Hexagonal System.

162. Choice of Axis. — The position of the vertical crystallographic axis is fixed in all the classes of this system since it coincides with the axis of hexagonal symmetry in the hexagonal division and that of trigonal symmetry in the rhombohedral division. The three horizontal axes are also fixed in direction except in the normal class and the subordinate hemimorphic class of the hexagonal division; in these there is a choice of two positions according to which of the two sets of vertical planes of symmetry is taken as the axial set.

163. Axial and Angular Elements. — The axial element is the length of the vertical axis, c, in terms of a horizontal axis, a; in other words, the axial ratio of $a : c$. A single measured angle (in any zone but the prismatic) may be taken as the fundamental angle from which the axial ratio can be obtained.

The angular element is usually taken as the angle between the base $c(0001)$ and the unit first order pyramid $(10\bar{1}1)$, that is, $0001 \wedge 10\bar{1}1$.

The relation between this angle and the axis c is given by the formula

$$\tan\,(0001 \wedge 10\bar{1}1) \times \frac{1}{2}\sqrt{3} = c.$$

The vertical axis is also easily obtained from the unit second order pyramid, since

$$\tan\,(0001 \wedge 11\bar{2}2) = c.$$

These relations become general by writing them as follows:

$$\tan\,(0001 \wedge h0\bar{h}l) \times \frac{1}{2}\sqrt{3} = \frac{h}{l} \times c;$$

$$\tan\,(0001 \wedge h{\cdot}h{\cdot}2\bar{h}{\cdot}l) = \frac{2h}{l} \times c.$$

In general it is easy to obtain any required angle between the poles of two faces on the spherical projection either by the use of the tangent (or cotangent) relation, or by the solution of spherical triangles, or by the application of both methods. In practice most of the triangles used in calculation are right-angled.

164. Tangent and Cotangent Relations. — The tangent relation holds good in any zone from $c(0001)$ to a face in the prismatic zone. For example:

$$\frac{\tan\,(0001 \wedge h0\bar{h}l)}{\tan\,(0001 \wedge 10\bar{1}1)} = \frac{h}{l}; \quad \frac{\tan\,(0001 \wedge h{\cdot}h{\cdot}2\bar{h}{\cdot}l)}{\tan\,(0001 \wedge 11\bar{2}2)} = \frac{2h}{l}.$$

In the prismatic zone, the cotangent formula takes a simplified form; for example, for a dihexagonal prism, $hk\bar{i}0$, as $(21\bar{3}0)$:

$$\cot\,(10\bar{1}0 \wedge hk\bar{i}0) = \frac{2h+k}{k}\sqrt{\frac{1}{3}};$$

$$\cot\,(11\bar{2}0 \wedge hk\bar{i}0) = \frac{h+k}{h-k}\sqrt{3}.$$

The sum of the angles $(10\bar{1}0 \wedge hki0)$ and $(11\bar{2}0 \wedge hk\bar{i}0)$ is equal to 30°.

Further, the last equations can be written in a more general form, applying to any pyramid $(hk\bar{i}l)$ in a zone, first between $10\bar{1}0$ and a face in the zone 0001 to $01\bar{1}0$, where the angle between $10\bar{1}0$ and this face is known; or again, for the same pyramid, in a zone between $11\bar{2}0$ and a face in the zone 0001 to $10\bar{1}0$, the angle between $11\bar{2}0$ and this face being given. For example (cf. Fig. 229, p. 99), if the first-mentioned zone is $10\bar{1}0{\cdot}hk\bar{i}l{\cdot}01\bar{1}1$ and the second is $11\bar{2}0{\cdot}hk\bar{i}l{\cdot}10\bar{1}1$, then

$$\cot\,(10\bar{1}0 \wedge hk\bar{i}l) = \cot\,(10\bar{1}0 \wedge 01\bar{1}1)\,.\,\frac{2h+k}{k},$$

and

$$\cot\,(11\bar{2}0 \wedge hk\bar{i}l) = \cot\,(11\bar{2}0 \wedge 10\bar{1}1)\,.\,\frac{h+k}{h-k}.$$

Also similarly for other zones,

$$\cot\,(10\bar{1}0 \wedge hk\bar{i}l) = \cot\,(10\bar{1}0 \wedge 02\bar{2}1)\,.\,\frac{2h+k}{k}, \text{ etc.}$$

$$\cot\,(11\bar{2}0 \wedge hk\bar{i}l) = \cot\,(11\bar{2}0 \wedge 20\bar{2}1)\cdot\frac{h+k}{h-k},\ \text{etc.}$$

165. Other Angular Relations. — The following simple relations are of frequent use:
(1) For a *hexagonal pyramid of the first order*,

$$\tan \tfrac{1}{2}\,(10\bar{1}1 \wedge 01\bar{1}1) = \sin \xi \sqrt{\tfrac{1}{3}},\quad \text{where } \tan \xi = c,$$

and in general

$$\tan \tfrac{1}{2}\,(h0\bar{h}l \wedge 0h\bar{h}l) = \sin \xi_{,} \sqrt{\tfrac{1}{3}},\quad \text{where } \tan \xi_{,} = \frac{h}{l}\,c.$$

(2) For a *hexagonal pyramid of the second order*, as $(11\bar{2}2)$,

$$2 \sin \tfrac{1}{2}\,(11\bar{2}2 \wedge \bar{1}2\bar{1}2) = \sin \xi, \qquad \text{and} \qquad \tan \xi = c.$$

(3) For a *rhombohedron*

$$\sin \tfrac{1}{2}\,(10\bar{1}1 \wedge \bar{1}101) = \sin \alpha \sqrt{\tfrac{3}{4}},\quad \text{where } \alpha = (0001 \wedge 10\bar{1}1);$$

in general

$$\sin \tfrac{1}{2}\,(h0\bar{h}l \wedge \bar{h}h0l) = \sin \alpha_{,} \sqrt{\tfrac{3}{4}},\ \text{where } \alpha_{,} = (0001 \wedge h0\bar{h}l).$$

166. Zonal Relations. — The zonal equations, described in Arts 45, 46, apply here as in other systems, only that it is to be noted that one of the indices referring to the horizontal axes, preferably the third, i, is to be dropped in the calculations and only the other three employed. Thus the indices (u, v, w) of the zone in which the faces $(hk\bar{i}l)$, $(pq\bar{r}t)$ lie are given by the scheme

$$\begin{array}{c|ccccc|c} h & k & l & h & k & & l \\ p & q & t & p & q & & t \end{array}$$

where $\qquad \text{u} = kt - lq, \qquad \text{v} = lp - ht, \qquad \text{w} = hq - kp.$

For example (Fig. 226) the face n lies in the zone mv, $10\bar{1}0 \cdot 21\bar{3}1$ and also in the zone au, $11\bar{2}0 \cdot 20\bar{2}1$. For the first zone the values obtained are: $u = 0$, $v = \bar{1}$, $w = 1$; for the second zone, $e = 1$, $f = \bar{1}$, $g = \bar{2}$. Combining these zone symbols according to the usual scheme

$$\begin{array}{c|ccccc|c} 0 & \bar{1} & 1 & 0 & \bar{1} & & 1 \\ 1 & \bar{1} & \bar{2} & 1 & \bar{1} & & \bar{2} \\ \hline & 3 & 1 & 1 & & & \end{array}$$

The face n has, therefore, the indices $31\bar{4}1$, since further $i = -(h + k)$.

167. Formulas. — The following formulas in which c equals the unit length of the vertical axis are sometimes useful:

(1) The distances (see Fig. 229) of the pole of any face $(hk\bar{i}l)$ from the poles of the faces $(10\bar{1}0)$, $(01\bar{1}0)$, $(\bar{1}100)$, and (0001) are given by the following equations,

$$\cos\,(hk\bar{i}l)\,(10\bar{1}0) = \frac{c\,(k+2h)}{\sqrt{3l^2 + 4c^2\,(h^2+k^2+hk)}}.$$

$$\cos\,(hk\bar{i}l)\,(01\bar{1}0) = \frac{c\,(2k+h)}{\sqrt{3l^2 + 4c^2\,(h^2+k^2+hk)}}.$$

$$\cos\,(hk\bar{i}l)\,(\bar{1}100) = \frac{c\,(h-k)}{\sqrt{3l^2 + 4c^2\,(h^2+k^2+hk)}}.$$

$$\cos\,(hkil)\,(0001) = \frac{l\sqrt{3}}{\sqrt{3l^2 + 4c^2\,(h^2+k^2+hk)}}.$$

(2) The distance (PQ) between the poles of any two faces $\mathrm{P}(hk\bar{i}l)$ and $\mathrm{Q}(pq\bar{r}t)$ is given by the equation

$$\cos PQ = \frac{3lt + 2c^2(hq + pk + 2hp + 2kq)}{\sqrt{[3l^2 + 4c^2(h^2 + k^2 + hk)]\,[3t^2 + 4c^2(p^2 + q^2 + pq)]}}.$$

(3) For special cases the above formula becomes simplified; it serves to give the value of the normal angles for the several forms in the system. They are as follows:

(*a*) *Pyramid of First Order* $(h0hl)$, Fig. 224:

$$\cos X\ (\text{terminal}) = \frac{3l^2 + 2h^2c^2}{3l^2 + 4h^2c^2};\quad \cos Z\ (\text{basal}) = \frac{4h^2c^2 - 3l^2}{3l^2 + 4h^2c^2}.$$

(*b*) *Pyramid of Second Order* $(h{\cdot}h{\cdot}2\bar{h}{\cdot}l)$, Fig. 225:

$$\cos Y\ (\text{terminal}) = \frac{l^2 + 2c^2h^2}{l^2 + 4c^2h^2};\quad \cos Z\ (\text{basal}) = \frac{4c^2h^2 - l^2}{l^2 + 4c^2h^2}.$$

(*c*) *Dihexagonal Pyramid* $(hk\bar{i}l)$:

$$\cos X\ (\text{see Fig. 226}) = \frac{3l^2 + 2c^2(h^2 + k^2 + 4hk)}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

$$\cos Y\ (\text{see Fig. 226}) = \frac{3l^2 + 2c^2(2h^2 + 2hk - k^2)}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

$$\cos Z\ (\text{basal}) = \frac{4c^2(h^2 + k^2 + hk) - 3l^2}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

(*d*) *Dihexagonal Prism* $(hk\bar{i}0)$, Fig. 222:

$$\cos X\ (\text{axial}) = \frac{h^2 + k^2 + 4hk}{2(h^2 + k^2 + hk)}.\qquad \cos Y\ (\text{diagonal}) = \frac{2h^2 + 2hk - k^2}{2(h^2 + k^2 + hk)}.$$

(*e*) *Rhombohedron* $(10\bar{1}1)$:

$$\cos X\ (\text{terminal}) = \frac{3l^2 - 2h^2c^2}{3l^2 + 4h^2c^2}.$$

(*f*) *Scalenohedron* $(hk\bar{i}l)$:

$$\cos X\ (\text{see Fig. 259}) = \frac{3l^2 + 2c(2k^2 + 2hk - h^2)}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

$$\cos Y\ (\text{see Fig. 259}) = \frac{3l^2 + 2c^2(2h^2 + 2hk - k^2)}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

$$\cos Z\ (\text{basal}) = \frac{2c^2(h^2 + k^2 + 4hk) - 3l^2}{3l^2 + 4c^2(h^2 + k^2 + hk)}.$$

168. Angles. — The angles for some commonly occurring dihexagonal prisms with the first and second order prisms are given in the following table:

	$m(1010)$	$a(11\bar{2}0)$
$51\bar{6}0$	8° 57′	21° 3′
$41\bar{5}0$	10 53½	19 6½
$31\bar{4}0$	13 54	16 6
$52\bar{7}0$	16 6	13 54
$21\bar{3}0$	19 6½	10 53½
$32\bar{5}0$	23 24¾	6 35¼
$54\bar{9}0$	26 19¾	3 40¼

169. The Miller Axes and Indices. The forms of the hexagonal system were referred by Miller to a set of three equal oblique axes which were taken parallel to the edges of the unit positive rhombohedron of the species. Fig. 291 represents such a rhombohedron with the position of the Miller axes shown. This choice of axes for hexagonal forms has the grave objection that in several cases the faces of the same form are represented by two sets of different indices; for example the faces of the pyramid of the first order would have the indices, $100, 22\bar{1}, 010, \bar{1}22, 001, 2\bar{1}2$. This objection, however, disappears if the

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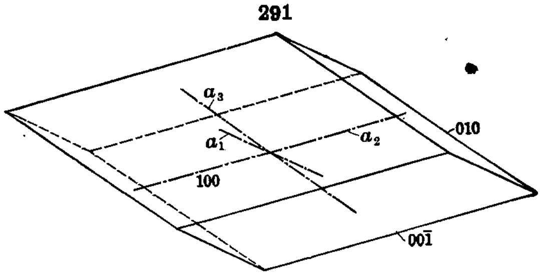

Miller axes and indices are used only for forms in the Rhombohedral Division, that is for forms belonging to classes which are characterized by a vertical axis of trigonal symmetry. It is believed, however, that the mutual relations of all the classes of both divisions of the hexagonal system among themselves (as also to the classes of the tetragonal system), both morphological and physical are best brought out by keeping throughout the same axes, namely those of Fig. 218, Art. **121**. The Miller method has, however, been adopted by a number of authors and consequently it is necessary to give the following brief description.

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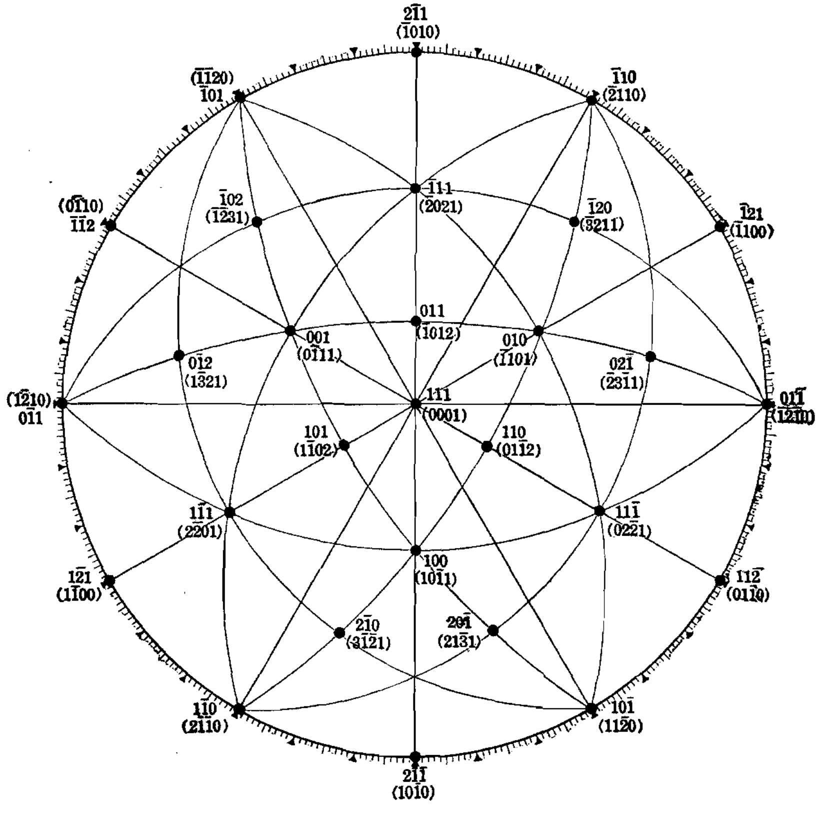

Miller and Miller-Bravais Indices Compared

Fig. 292 shows in stereographic projection the common hexagonal-rhombohedral forms with their Miller indices and in parentheses the corresponding indices when the faces are referred to the four axial system. It will be noted that the faces of the unit positive rhombohedron have the indices 100, 010, and 001 and those of the negative unit rhombohedron have $22\bar{1}$, $\bar{1}22$, $2\bar{1}2$. These two forms together give the faces of the hexagonal pyramid of the first order (see above). The hexagonal prism of the first order is represented by $2\bar{1}\bar{1}$, etc., while the second order prism has $10\bar{1}$, etc. The dihexagonal pyramid has also two sets of indices (*hkl*) and (*efg*); of these the symbol (*hkl*) belongs to the positive scalenohedron and (*efg*) to the negative form. In this as in other cases it is true that $e = 2h + 2k - l$, $f = 2h - k + 2l$, $g = -h + 2k + 2l$. For example, the faces of the form $20\bar{1}$, etc., belong in the Rhombohedral Division of this system to the scalenohedron $(21\bar{3}1)$ while the complementary negative form would have the indices $52\bar{4}$, etc.

The relation between the Miller-Bravais and the Miller indices for any form can be obtained from the following expression, where $(hkil)$ represents the first and (pqr) the second.

$$\frac{h}{p-q}=\frac{k}{p-r}=\frac{i}{r-p}=\frac{l}{p+q+r}.$$

The relation between the Miller indices for hexagonal forms and those of isometric forms should be noted. If we conceive of the isometric cube as a rhombohedron with interfacial angles of 90° and change the orientation so that the normal to the octahedral face (111) becomes vertical we get a close correspondence between the two. This will be seen by a comparison of the two stereographic projections, Figs. 292 and 125.

170. To determine, by plotting, the length of the vertical axis of a hexagonal mineral, given the position on the stereographic projection of the pole of a face with known indices. To illustrate this problem it is assumed that the mineral in question is beryl and that the position of the pole $p(10\bar{1}1)$ is known, Fig. 293. Let the three lines a_1, a_2, a_3 represent the horizontal axes with their unit lengths equalling the radius of the circle. Draw a line from the center of the projection through the pole p. Draw another line (which will be at right angles to the first) joining the ends of a_1 and $-a_3$. This will be parallel to a_2 and will represent the intercept of $p(10\bar{1}1)$ upon the plane of the horizontal axes. In order to plot the intercept of p upon the vertical axis construct in the upper left-hand quadrant of the figure a right-angle triangle the base of which shall be equal to O–P, the vertical side of which shall represent the c axis and the hypothenuse shall show the slope of the face and give its intercept upon the c axis. The direction of the hypothenuse is determined by locating the normal to p from the angle measured from the center of the projection to its pole. Since the face has been assumed to have an unit intercept on the vertical axis the distance O–M, which equals 0·49 (in terms of the length of the horizontal axes, which equals 1·00), gives the unit length of the c axis for beryl.

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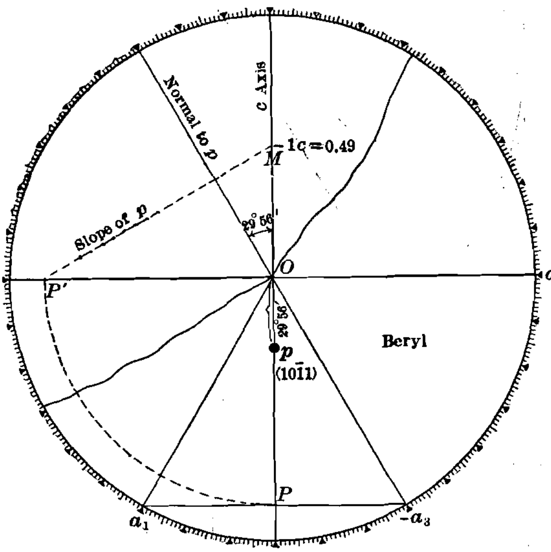

Determination of unit length of c axis, having given the position of $p(10\bar{1}1)$

171. To determine the indices of a face of a hexagonal form of a known mineral, given the position of its pole on the stereographic projection. In Fig. 294 it is assumed that the position of the pole v of a crystal face on calcite is known. To determine its indices, first draw a radial line through the pole and then erect a perpendicular to it, starting the line from the end of one of the horizontal axes. This line will represent the direction of the intersection of the crystal face with the horizontal plane and its relative intercepts on the horizontal axes will give the first three numbers of the parameters of the face, namely $1a_1$, $2a_2$, $\frac{2}{3}-a_3$. To determine the relative intercept on the c axis transfer the distance O–P to the upper left-hand quadrant of the figure, then having measured the angular distance between the center of the projection and v by means of the stereographic protractor draw the pole to the face in the proper position. Draw then a line at right angles to this pole starting from the point P'. This line gives the intercept of the face upon the line representing the vertical axis. In this case the intercept has a value of 1·7 when the length of the horizontal axes is taken as equal to 1·0. This distance 1·7 is seen to be twice the unit length of the c axis for calcite, 0·85. Therefore the parameters of the face in question upon the four axes are $1a_1$, $2a_2$, $\frac{2}{3}-a_3$, $2c$, which give $21\bar{3}1$ for the indices of the face v.

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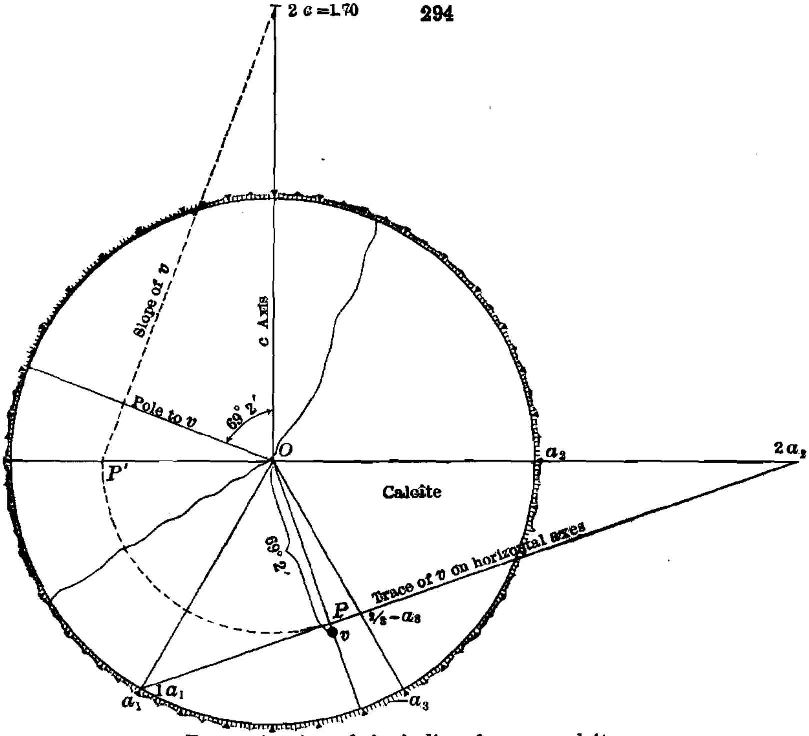

Determination of the indices for *v* on calcite

172. To determine, by plotting, the indices of hexagonal forms, given the position of their poles on the gnomonic projection. To illustrate this problem one sectant of the gnomonic projection of the important forms of beryl, Fig. 228, is reproduced in Fig. 295. The directions of the three horizontal axes, a_1, a_2 and a_3 are indicated by the heavy lines. From the poles of the faces perpendiculars are drawn to these three axes. It will be noted that the various intercepts made upon the axes by these lines have simple rational relations to each other. One of these intercepts is chosen as having the length of 1 (this length will be equivalent to the unit length of the *c* crystallographic axis, see below) and the others are then given in terms of it.

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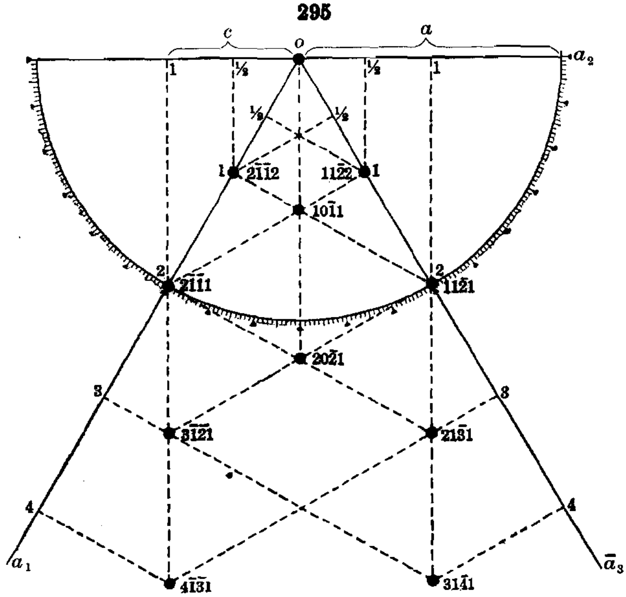

The indices of each face are obtained directly by taking these intercepts upon the three horizontal axes in their proper order and by adding a 1 as the fourth figure. If necessary clear of fractions, as in the case of the second order pyramid, $11\bar{2}2$.

173. To determine the axial ratio of a hexagonal mineral from the gnomonic projection of its forms. The gnomonic projection of the beryl forms, Fig. 295, may be used as an illustrative example. The radius of the fundamental circle, *a*, is taken as equal to the length of the horizontal axes and is given a value of 1. Then the length of the fundamental intercept of the lines dropped perpendicularly from the poles, *i.e.* the distance *c*, will equal the length of the *c* axis when expressed in terms of the length of *a*. In the case of beryl this ratio is $a : c = 1{\cdot}00 : 0{\cdot}499$. That this relationship is true can be proved in the same manner as in the case of the tetragonal system, see Art. **117**, p. 93.

IV. ORTHORHOMBIC SYSTEM

(Rhombic or Prismatic System)

174. Crystallographic Axes. — The *orthorhombic system* includes all the forms which are referred to three axes at right angles to each other, all of different lengths.

Any one of the three axes may be taken as the vertical axis, *c*. Of the two horizontal axes the longer is always taken as the *b* or macro-axis * and when orientated is parallel to the observer. The *a* or brachy-axis is the shorter of the two horizontal axes and is perpendicular to the observer. The length of the *b* axis is taken as unity and the lengths of the other axes are expressed in terms of it. The axial ratio for barite, for instance, is $a : b : c = 0{\cdot}815 : 1{\cdot}00 : 1{\cdot}31$. Fig. 296 shows the crystallographic axes for barite.

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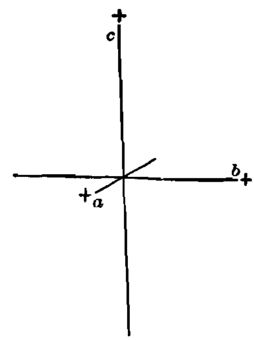

Orthorhombic Axes (Barite)

1. NORMAL CLASS (25). BARITE TYPE

(Orthorhombic Bipyramidal or Holohedral Class)

175. Symmetry. — The forms of the *normal class* of the orthorhombic system are characterized by three axes of binary symmetry, which directions are coincident with the crystallographic axes. There are also three unlike planes of symmetry at right angles to each other in which lie the crystallographic axes.

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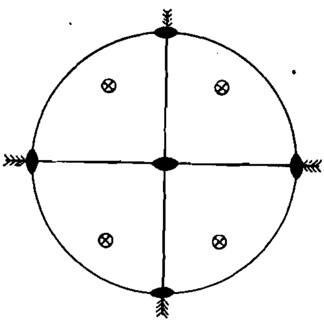

Symmetry of Normal Class Orthorhombic System

The symmetry of the class is exhibited in the accompanying stereographic projection, Fig. 297. This should be compared with Fig. 91 (p. 53) and Fig. 167 (p. 77), representing the symmetry of the normal classes of the isometric and tetragonal systems respectively. It will be seen that while normal isometric crystals are developed alike in the three axial directions, those of the tetragonal type have a like development only in the direction of the two horizontal axes, and

* The prefixes *brachy-* and *macro-* used in this system (and also in the triclinic system) are from the Greek words, *βραχύς, short,* and *μακρός, long.*